

An asymptotic formula of the divergent bilateral basic hypergeometric series

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Abstract

We show an asymptotic formula of the divergent bilateral basic hypergeometric series ${}_1\psi_0(a; -; q, \cdot)$ with using the q -Borel-Laplace method. We also give the limit $q \rightarrow 1 - 0$ of our asymptotic formula.

1 Introduction

In this paper, we show an asymptotic formula of the divergent bilateral basic hypergeometric series

$${}_1\psi_0(a; -; q; x) := \sum_{n \in \mathbb{Z}} (a; q)_n \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{-1} x^n.$$

Here, $(a; q)_n$ is the q -shifted factorial;

$$(a; q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq) \cdots (1-aq^{n-1}), & n \geq 1, \\ [(1-aq^{-1})(1-aq^{-2}) \cdots (1-aq^n)]^{-1}, & n \leq -1 \end{cases}$$

moreover, $(a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n$ and

$$(a_1, a_2, \dots, a_m; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty.$$

The q -shifted factorial $(a; q)_n$ is a q -analogue of the shifted factorial

$$(\alpha)_n = \alpha \{\alpha + 1\} \cdots \{\alpha + (n-1)\}.$$

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The series ${}_1\psi_0(a; -, q, \cdot)$ is related to the bilateral basic hypergeometric series ${}_1\psi_1(a; b; q, \cdot)$. At first, we review the series

$${}_1\psi_1(a; b; q, z) := \sum_{n \in \mathbb{Z}} \frac{(a; q)_n}{(b; q)_n} z^n.$$

This series has Ramanujan's summation formula [3]

$${}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1, \quad (1)$$

which was first given by Ramanujan [4]. This formula is considered as an extension of the q -binomial theorem [3, 1];

$$\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.$$

We also regard the summation (1) as a q -analogue of the bilateral binomial theorem [6] discovered by M. E. Horn. If α and β are complex numbers $\Re(\beta - \alpha) > 1$ and z is a complex number with $|z| = 1$, then

$$\sum_{n \in \mathbb{Z}} \frac{(\alpha)_n}{(\beta)_n} z^n = \frac{(1 - z)^{\beta - \alpha - 1}}{(-z)^{\beta - 1}} \frac{\Gamma(1 - \alpha)\Gamma(\beta)}{\Gamma(\beta - \alpha)}. \quad (2)$$

We remark that the limit $q \rightarrow 1 - 0$ of (1) with suitable condition gives the bilateral binomial theorem (2).

The series ${}_1\psi_1(a; b; q, z)$ satisfies the following q -difference equation

$$\left(\frac{b}{q} - az\right) u(qz) + (z - 1)u(z) = 0.$$

In this paper, we study the degeneration of this equation as follows;

$$\left(\frac{1}{q} - ax\right) \tilde{u}(qx) + x\tilde{u}(x) = 0.$$

This equation has a formal solution

$$\tilde{u}(x) = {}_1\psi_0(a; -, q, x). \quad (3)$$

But $\tilde{u}(x)$ is a divergent around the origin [3] and its properties are not clear. In section three, we show an asymptotic formula of the series (3) with using the q -Borel-Laplace transformations. The q -Borel-Laplace transformations are introduced in the study of connection problems on linear q -difference equations between the origin and the infinity.

Connection problems on q -difference equations with regular singular points are studied by G. D. Birkhoff [2]. The first example of the connection formula is given by G. N. Watson [13] in 1910 as follows;

$$\begin{aligned} {}_2\varphi_1(a, b; c; q, x) &= \frac{(b, c/a; q)_\infty (ax, q/ax; q)_\infty}{(c, b/a; q)_\infty (x, q/x; q)_\infty} {}_2\varphi_1(a, aq/c; aq/b; q, cq/abx) \\ &+ \frac{(a, c/b; q)_\infty (bx, q/bx; q)_\infty}{(c, a/b; q)_\infty (x, q/x; q)_\infty} {}_2\varphi_1(b, bq/c; bq/a; q, cq/abx). \end{aligned} \quad (4)$$

The function ${}_2\varphi_1(a, b; c; q, x)$ is the basic hypergeometric series

$${}_2\varphi_1(a, b; c; q, x) = \sum_{n \geq 0} \frac{(a, b; q)_n}{(c; q)_n (q; q)_n} x^n,$$

which is introduced by E. Heine [5]. This series satisfies the second order linear q -difference equation

$$(c - abqx)u(q^2x) - \{c + q - (a + b)qx\}u(qx) + q(1 - x)u(x) = 0. \quad (5)$$

around the origin. On the other hand, the equation (5) has a fundamental system of solutions

$$v_1(x) = \frac{\theta(-ax)}{\theta(-x)} {}_2\varphi_1\left(a, \frac{aq}{c}; \frac{aq}{b}; q, \frac{cq}{abx}\right), \quad v_2(x) = \frac{\theta(-bx)}{\theta(-x)} {}_2\varphi_1\left(b, \frac{bq}{c}; \frac{bq}{a}; q, \frac{cq}{abx}\right)$$

around the infinity [14]. Here, $\theta(\cdot)$ is the theta function of Jacobi (see section two). In the connection formula (4), we remark that the connection coefficients are given by q -elliptic functions.

Other connection formulae, especially connection formulae of irregular singular type q -special functions has not known for a long time. Recently, C. Zhang gives some connection formulae by two different types of q -Borel transformations and q -Laplace transformations [14, 15, 16], which are introduced by J. Sauloy[11]. We assume that $f(x)$ is a formal power series $f(x) = \sum_{n \geq 0} a_n x^n$, $a_0 = 1$.

Definition 1. For any power series $f(x)$, the q -Borel transformation of the first kind \mathcal{B}_q^+ is

$$(\mathcal{B}_q^+ f)(\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n =: \varphi(\xi).$$

For an entire function φ , the q -Laplace transformation of the first kind $\mathcal{L}_{q,\lambda}^+$ is

$$(\mathcal{L}_{q,\lambda}^+ \varphi)(x) := \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta\left(\frac{\lambda q^n}{x}\right)}, \quad \lambda \notin q^{\mathbb{Z}}.$$

Similarly, the q -Borel transformation of the second kind \mathcal{B}_q^- is

$$(\mathcal{B}_q^- f)(\xi) := \sum_{n \geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n =: g(\xi)$$

and the q -Laplace transformation of the second kind \mathcal{L}_q^- is

$$(\mathcal{L}_q^- g)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi) \theta\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}.$$

We remark that each q -Borel transformation is a formal inverse of each q -Laplace transformation;

$$\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f, \quad \mathcal{L}_q^- \circ \mathcal{B}_q^- f = f.$$

We can find applications of the q -Borel-Laplace method of the first kind in [14, 15, 10]. This summation method is powerful to deal with divergent type basic hypergeometric series and to study the q -Stokes phenomenon. Applications of the method of the second kind can be found in [16, 8, 9, 10]. But other examples, for example, applications to bilateral basic hypergeometric series, has not known.

In this paper, we study an asymptotic formula of the divergent bilateral basic hypergeometric series ${}_1\psi_0(a; -; q, \cdot)$ from viewpoint of connection problems on linear q -difference equations. We show the following theorem.

Theorem. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have

$$\hat{{}_1\psi_0}(a; -; \lambda; q, x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda)}{\theta(q\lambda)} \frac{\theta(ax/\lambda)}{\theta(x/\lambda)} \frac{1}{(1/ax; q)_\infty},$$

where $1 < |ax|$.

Here, the function ${}_1\hat{\psi}_0(a; -; \lambda; q, x)$ is the q -Borel-Laplace transform of the series ${}_1\psi_0(a; -; q, x)$ and the set $[-\lambda; q]$ is the q -spiral which is given by $[\lambda; q] := \{\lambda q^k : k \in \mathbb{Z}\}$. We also show the limit $q \rightarrow 1 - 0$ of our asymptotic formula in section four. If we take a suitable limit $q \rightarrow 1 - 0$ of the theorem above, we formally obtain the asymptotic formula of the bilateral hypergeometric series ${}_1H_0$.

2 Basic notations

In this section, we fix our notations. We assume that $0 < |q| < 1$ and σ_q is the q -difference operator such that $\sigma_q f(x) = f(qx)$. The basic hypergeometric series is

$${}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n (q; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \quad (6)$$

The radius of convergence ρ of the basic hypergeometric series is given by [7]

$$\rho = \begin{cases} \infty, & \text{if } r < s + 1, \\ 1, & \text{if } r = s + 1, \\ 0, & \text{if } r > s + 1. \end{cases}$$

The bilateral basic hypergeometric series is

$${}_r\psi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, x) := \sum_{n \in \mathbb{Z}} \frac{(a_1, \dots, a_r; q)_n}{(b_1, \dots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{s-r} x^n. \quad (7)$$

If $s < r$, the series (7) diverges for $x \neq 0$ and if $r = s$, the series (7) converges $|b_1 \dots b_s / a_1 \dots a_r| < |x| < 1$ (see [3] for more detail). The series (6) is a q -analogue of the generalized hypergeometric function

$${}_rF_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; x) := \sum_{n \geq 0} \frac{(\alpha_1, \dots, \alpha_r)_n}{(\beta_1, \dots, \beta_s)_n n!} x^n$$

and the series (7) is a q -analogue of the bilateral hypergeometric function

$${}_rH_s(\alpha_1, \dots, \alpha_r; \beta_1, \dots, \beta_s; x) := \sum_{n \in \mathbb{Z}} \frac{(\alpha_1, \dots, \alpha_r)_n}{(\beta_1, \dots, \beta_s)_n} x^n.$$

By D'Alembert's ratio test, it can be checked that ${}_rH_r$ converges only for $|x| = 1$ [12], provided that $\Re(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_r) > 1$.

The q -exponential function $e_q(x)$ is

$$e_q(x) := {}_1\varphi_0(0; -; q, x) = \sum_{n \geq 0} \frac{x^n}{(q; q)_n}$$

The function $e_q(x)$ has the infinite product representation as follows;

$$e_q(x) = \frac{1}{(x; q)_\infty}, \quad |x| < 1.$$

The limit $q \rightarrow 1 - 0$ of $e_q(x)$ is the exponential function [3];

$$\lim_{q \rightarrow 1-0} e_q(x(1-q)) = e^x. \quad (8)$$

The q -gamma function $\Gamma_q(x)$ is

$$\Gamma_q(x) := \frac{(q; q)_\infty}{(q^x; q)_\infty} (1-q)^{1-x}, \quad 0 < q < 1.$$

The limit $q \rightarrow 1 - 0$ of $\Gamma_q(x)$ gives the gamma function [3];

$$\lim_{q \rightarrow 1-0} \Gamma_q(x) = \Gamma(x). \quad (9)$$

The theta function of Jacobi is given by

$$\theta(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \quad \forall x \in \mathbb{C}^*.$$

Jacobi's triple product identity is

$$\theta(x) = (q, -x, -q/x; q)_\infty$$

and the theta function satisfies the following q -difference equation

$$\theta(q^k x) = x^{-k} q^{-\frac{k(k-1)}{2}} \theta(x), \quad \forall k \in \mathbb{Z}.$$

The inversion formula is

$$\theta(x) = x\theta(1/x). \quad (10)$$

We remark that $\theta(\lambda q^k/x) = 0$ if and only if $x \in [-\lambda; q]$. In our study, the following proposition [15] is useful to consider the limit $q \rightarrow 1 - 0$ of our formula.

Proposition 1. *For any $x \in \mathbb{C}^*(-\pi < \arg x < \pi)$, we have*

$$\lim_{q \rightarrow 1-0} \frac{\theta(q^\alpha x)}{\theta(q^\beta x)} = x^{\beta-\alpha} \quad (11)$$

and

$$\lim_{q \rightarrow 1-0} \frac{\theta\left(\frac{q^\alpha x}{(1-q)}\right)}{\theta\left(\frac{q^\beta x}{(1-q)}\right)} (1-q)^{\beta-\alpha} = \left(\frac{1}{x}\right)^{\alpha-\beta}. \quad (12)$$

We also use the limit [3] as follows;

$$\lim_{q \rightarrow 1-0} \frac{(xq^\alpha; q)_\infty}{(x; q)_\infty} = (1-x)^{-\alpha}, \quad |x| < 1. \quad (13)$$

3 An asymptotic formula of the divergent series ${}_1\psi_0(a; -; q, x)$

In this section, we show an asymptotic formula of the divergent series ${}_1\psi_0(a; -; q, x)$. At first, we review Ramanujan's sum for ${}_1\psi_1(a; b; q, z)$ and its property. Ramanujan gives the following sum [4]

$${}_1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1. \quad (14)$$

We can regard this sum as a q -analogue of the bilateral binomial theorem [6] as follows;

Theorem. (Horn [6]) *If z is a complex number such that $|z| = 1$, we have*

$${}_1H_1(\alpha; \beta; z) = \frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha)} \frac{(1-z)^{\beta-\alpha-1}}{(-z)^{\beta-1}}, \quad (15)$$

provided that $\Re(\beta - \alpha) > 1$.

Let z is a complex number such that $-\pi < \arg z < \pi$. The summation (14) is rewritten by

$$\frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} = \frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty} \frac{\theta(-az)}{\theta(-aqz/b)} \frac{(aqz/b; q)_\infty}{(z; q)_\infty}. \quad (16)$$

In (16), we put $a = q^\alpha$ and $b = q^\beta$ (with $\Re(\beta - \alpha) > 1$), we obtain

$${}_1\psi_1(q^\alpha; q^\beta; q, z) = \frac{\Gamma_q(\beta)\Gamma_q(1-\alpha)}{\Gamma_q(\beta-\alpha)} \frac{\theta(q^\alpha(-z))}{\theta(q^{\alpha+1-\beta}(-z))} \frac{(q^{\alpha+1-\beta}z; q)_\infty}{(z; q)_\infty}, \quad (17)$$

where $|q^{\beta-\alpha}| < |z| < 1$. Combining (9), (11) and (13), we can check out that the limit $q \rightarrow 1-0$ of (17) gives the bilateral binomial theorem (15), provided that $-\pi < \arg z < \pi$.

The bilateral basic hypergeometric series ${}_1\psi_1(a; b; q, z)$ satisfies the q -difference equation

$$\left(\frac{b}{q} - az\right)u(qz) + (z-1)u(z) = 0. \quad (18)$$

We consider the degeneration of the equation (18) in the next section.

3.1 Local solutions of the degenerated equation

In the equation (18), we put $z = bx$ and take the limit $b \rightarrow \infty$, we obtain the equation

$$\left\{\left(\frac{1}{q} - ax\right)\sigma_q + x\right\}\tilde{u}(x) = 0. \quad (19)$$

The formal solution of (19) is

$$\tilde{u}(x) = {}_1\psi_0(a; -; q, x), \quad (20)$$

which is divergent around the origin. We consider “the basic hypergeometric type” solution around the infinity. We assume that the solution around the infinity is given by

$$\tilde{u}_\infty(x) := \frac{\theta(ax)}{\theta(x)}v_\infty(x) = \frac{\theta(ax)}{\theta(x)}\sum_{n \geq 0} v_n x^{-n}, \quad v_0 = 1.$$

By the q -difference equation of the theta function, we obtain the equation

$$\left\{\left(\frac{1}{aq} - x\right)\sigma_q + x\right\}v_\infty(x) = 0.$$

We remark that the function $\theta(ax)/\theta(x)$ satisfies the following q -difference equation

$$u(qx) = q^{-\alpha}u(x)$$

which is also satisfied by the function $u(x) = x^{-\alpha}$ with $\log_q a = \alpha$. We can check out that the series $v_\infty(x)$ is

$$v_\infty(x) = e_q \left(\frac{1}{ax} \right) = \frac{1}{(1/ax; q)_\infty}, \quad \left| \frac{1}{ax} \right| < 1.$$

Therefore, one of the solution of (19) around the infinity is

$$\tilde{u}_\infty(x) = \frac{\theta(ax)}{\theta(x)} e_q \left(\frac{1}{ax} \right). \quad (21)$$

In the following section, we study the relation between these solutions (20) and (21) with using the q -Borel-Laplace method of the first kind.

Remark 1. *We remark that “the bilateral basic hypergeometric type solution” of (19) around the infinity is given by*

$$w_\infty(x) = {}_1\psi_1 \left(0; \frac{q}{a}; q, \frac{1}{ax} \right) = \sum_{n \in \mathbb{Z}} \frac{1}{(q/a; q)_n} \left(\frac{1}{ax} \right)^n.$$

But this solution is not suitable for our argument. We choose and deal with the solution (21).

3.2 An asymptotic formula

In this section, we show an asymptotic formula of (20) with using the q -Borel-Laplace transformations of the first kind. We show the following theorem.

Theorem 1. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$, we have*

$${}_1\hat{\psi}_0(a; -; \lambda; q, x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda)}{\theta(q\lambda)} \frac{\theta(ax/\lambda)}{\theta(x/\lambda)} \frac{1}{(1/ax; q)_\infty},$$

where $1 < |ax|$.

Proof. We apply the q -Borel transformation \mathcal{B}_q^+ to the series ${}_1\psi_0(a; -; q, x)$. Then,

$$\psi(\xi) := (\mathcal{B}_q^+ {}_1\psi_0)(\xi) = {}_1\psi_1(a; 0; q, -\xi).$$

By Ramanujan’s sum (14), we obtain the infinite product representation of $\psi(\xi)$ as follows;

$$\psi(\xi) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(a\xi)}{\theta(\xi)} (-q/\xi; q)_\infty.$$

We apply the q -Laplace transformation $\mathcal{L}_{q,\lambda}^+$ to $\psi(\xi)$.

$$\begin{aligned}
(\mathcal{L}_{q,\lambda}^+ \psi)(x) &= \sum_{n \in \mathbb{Z}} \frac{\psi(\lambda q^n)}{\theta\left(\frac{\lambda q^n}{x}\right)} = \frac{(q; q)_\infty}{(q/a; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{1}{\theta\left(\frac{\lambda q^n}{x}\right)} \frac{\theta(a \lambda q^n)}{\theta(\lambda q^n)} \left(-\frac{q}{\lambda q^n}; q\right)_\infty \\
&= \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{1}{\theta(\lambda/x)} \frac{\theta(a\lambda)}{\theta(\lambda)} \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} \left(\frac{\lambda}{ax}\right)^n \left(-\frac{q}{\lambda q^n}; q\right)_\infty \\
&= \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{1}{\theta(\lambda/x)} \frac{\theta(a\lambda)}{\theta(\lambda)} (-q/\lambda; q)_{\infty 1} \psi_1(-\lambda; 0; q, 1/ax). \quad (22)
\end{aligned}$$

We remark that

$${}_1\psi_1(-\lambda; 0; q, 1/ax) = \frac{\theta(\lambda/ax)}{(-q/\lambda; q)_\infty (1/ax; q)_\infty}.$$

Combining (22) and (10), we obtain the conclusion. \square

Theorem 1 is rewritten by

$$(\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ \tilde{u})(x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda)}{\theta(q\lambda)} \frac{\theta(ax/\lambda)}{\theta(x/\lambda)} \frac{\theta(x)}{\theta(ax)} \tilde{u}_\infty(x).$$

We define the function $C(x; q)$ as follows;

$$C(x; q) := \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda)}{\theta(q\lambda)} \frac{\theta(ax/\lambda)}{\theta(x/\lambda)} \frac{\theta(x)}{\theta(ax)}.$$

Then, $C(x; q)$ is single valued as a function of x .

Corollary 1. *The function $C(x; q)$ is the q -elliptic function, i.e., $C(x; q)$ satisfies*

$$C(qx; q) = C(x; q), \quad C(e^{2\pi i} x; q) = C(x; q).$$

4 The limit $q \rightarrow 1 - 0$ of the asymptotic formula

In this section, we give the limit $q \rightarrow 1 - 0$ of our asymptotic formula. We assume that $x \in \mathbb{C}^* \setminus [-\lambda; q]$ ($-\pi < \arg x < \pi$). We put $a = q^\alpha$ and $x \mapsto x/(1 - q)$ in theorem 1 to consider the limit. We remark that the limit of the left-hand side in theorem 1 formally gives the bilateral hypergeometric series ${}_1H_0(\alpha; -; -x)$. We show the following theorem.

Theorem 2. For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$ ($-\pi < \arg x < \pi$), we have

$$\lim_{q \rightarrow 1-0} \frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \frac{\theta\left(\frac{q^\alpha x}{(1-q)\lambda}\right)}{\theta\left(\frac{x}{(1-q)\lambda}\right)} \frac{1}{\left((1-q)\frac{q^{-\alpha}}{x}; q\right)_\infty} = \frac{\Gamma(1-\alpha)}{x^\alpha} e^{\frac{1}{x}},$$

where $1 < |x|$.

We give the proof of theorem 2.

Proof. The right-hand side of theorem 1 is rewritten by

$$\begin{aligned} & \frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \frac{\theta\left(\frac{q^\alpha}{(1-q)} \frac{x}{\lambda}\right)}{\theta\left(\frac{1}{(1-q)} \frac{x}{\lambda}\right)} \frac{1}{\left((1-q)\frac{q^{-\alpha}}{x}; q\right)_\infty} \\ &= \Gamma_q(1-\alpha) \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \left\{ \frac{\theta\left(\frac{q^\alpha}{(1-q)} \frac{x}{\lambda}\right)}{\theta\left(\frac{q^0}{(1-q)} \frac{x}{\lambda}\right)} (1-q)^{-\alpha} \right\} e_q\left((1-q)\frac{1}{q^\alpha x}\right). \end{aligned}$$

Combining (9), (11), (12) and (8), we obtain the conclusion. \square

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